Geometry of Space, Time and Other Things

The Mathematics of Fiber Bundles

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Fiber Bundles

- Central to physics: classical mechanics, electrodynamics, quantum field theory, gravitation, superconductivity.
- It was not always that way!
- Unified (pseudo-)Riemannian Geometry (i.e. Gravitation) with Symplectic Geometry (classical mechanics) with Electrodynamcis with Yang-Mills theory with Superconductivity with Fermions (QFT)
- A single, unified framework for (almost) all of the fundamental theories of physics.
- And that is the topic today.

Forumlas and Intuition

Zen Koans

- There will will be a lot of equations today
- More than several semesters worth ...
- Notation is KEY: commonplace, widespread notation
- What does those formulas MEAN? Intuitively ??
- Interpretation of poetry, jokes of Zen koans
- Inutition alone is FAULTY. Formulas are PRECISE!
- Equations are tie-breakers for intuitive ideas
- Creativity and imagination are KEY
- It will be dizzying

Tee-shirt Equations

Before fiber bundles, it was a hot mess:

- Classical mechanics was Hamilton's equations

$$\dot{p} = -\frac{dH}{dq} \qquad \dot{q} = \frac{dH}{dp}$$

- Electrodynamics was Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad \vec{\nabla} \times \vec{B} = 4\pi\vec{j} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} = 0$$

- Gravitation was Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Superconductivity was the Ginzberg–Landau equations

$$\mathscr{L} = \alpha \left|\phi\right|^2 + \beta \left|\phi\right|^4 + \frac{1}{2m} \left| \left(-i\hbar\vec{\nabla} - 2e\vec{A} \right) \phi \right|^2 + \frac{\left|\vec{B}\right|^2}{2}$$

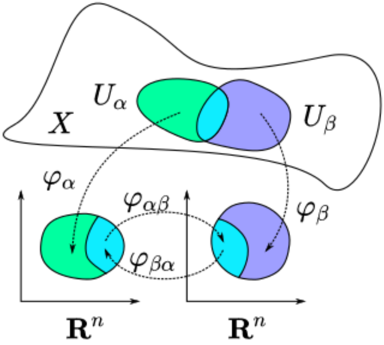
- Standard Model = Yang-Mills + Higgs + Fermions

Outline

- Manifold M as gluing of $\mathbb{R}^n$ - coordinate charts
- (Integrable) vector fields as hair/fur that can be combed
- Tangent vector space T_pM
- Back to basics: Vector spaces; notation: e_n as basis vector
- A frame field as $e_n(p)$ varying from point to point p.
- Frame fields can twist around, rotate, swirl.
- The rotation matrix A. The connection $A_i = \Gamma_{ij}^{\ \ k}$ aka Christoffel symbol
- Rotations & rotation matrices in 3D
- Curvature as total rotation after walking a loop.
- Parallel transport
- Geodesics

An atlas is:

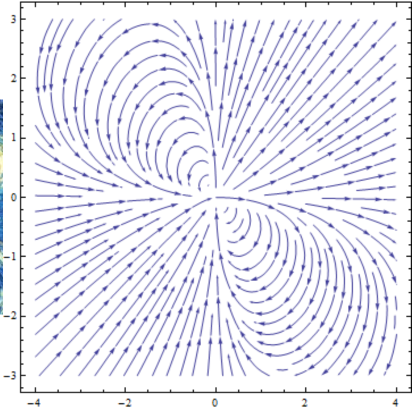
- A collection of regions U_α
- A collection of charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$
- A collection of "transition functions" $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$



Vector Fields

A vector field is:

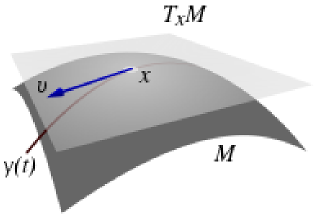
- A collection of vectors $\vec{v}_p$
- One for each point $p \in U_\alpha$
- Smooth, differentiable, integrable

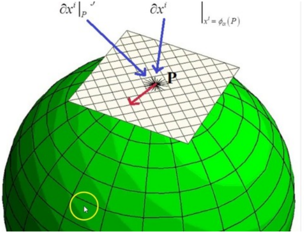


Tangent vector spaces

The tangent vector space T_pM is:

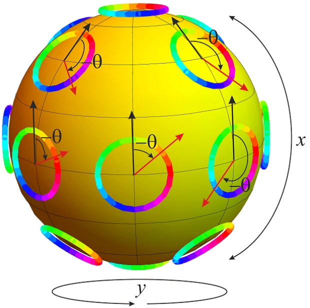
- A point $p \in U_\alpha$(that is, a point in $p \in M$)
- The collection of ALL possible vectors $\vec{v}_p \in T_pM$

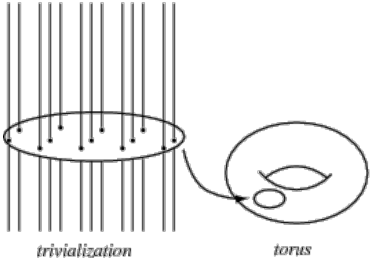




Tangent bundles - Fiber bundles

- The tangent bundle TM is the set of all T_pM for all $p \in M$
- The sphere bundle SM is a set of spheres S_pM, one for each $p \in M$
- The circle bundle is a set of circles, one for one for each $p \in M$
- The fiber bundle E is a set of fibers F, one for one for each $p \in M$



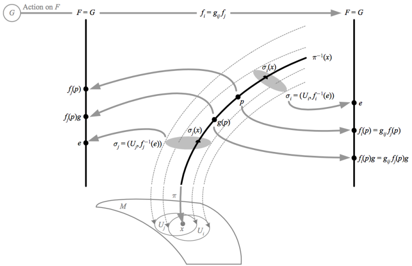


trivialization of a line bundle on a torus

Fiber Bundles

Properties of Fiber bundles

- Locally, they are trivial products $U_\alpha \times F$ of a chart U_α and a fiber F
- Neighboring fibers need to be glued (soldered) together; the connection!
- Works best when fibers have some natural symmetry
- A group G that moves you up and down a fiber F

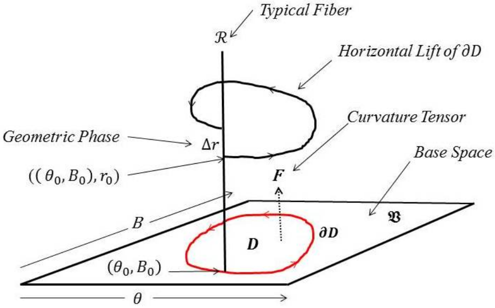


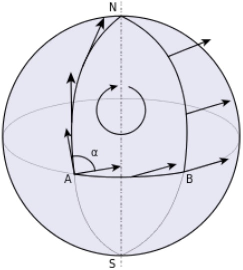
$$
\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\
U &
\end{tikzcd}
$$

Horizontal and Vertical Bundles

Gluing together neighboring fibers allows:

- Movement (horizontally) from fiber to fiber
- While carrying a coordinate frame (parallel transport)
- Closed paths in horizontal (base) space typically DON'T close on the bundle!
- That is, curvature!





Unifying Principle

Fiber bundles in Physics

- Circle bundles – U(1) – Electromagnetism
- Frame bundles – GL(n,R) – General Relativity (Reimannian geometry)
- Lie groups – SU(3) – Quarks & Gluons (strong force)
- Lie groups – SU(2) – Weak force (radioactive decay)
- Tangent bundles – Position and Momenta – Classical Mechanics (Symplectic geometry)
- Spinor bundles – Fermions
- Fischer Information (Kullback-Leibler divergence) – Quantum Mechanics

Interlocking tools

All fiber bundles have

- Horizontal and Vertical subspaces
- A connnection one-form (Christoffel symbols)
- Geodesics (shortest paths)
- Parallel transport (carrying around a coordinate frame)
- Curvature two-form (curvature tensor)

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Affine bundles have

- Solder form (canonical one-form)
- Torsion and Contorsion tensors

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Metric bundles have

- A metric
- Ricci and scalar curvature

Start at the begining

Back to basics

- Groups
- Actions
- Vectors
- Rotations
- Infinitessimal rotations (generators)
- Derivatives

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Advanced topics

- Differential forms
- Covariant derviative
- Curvature
- Torsion

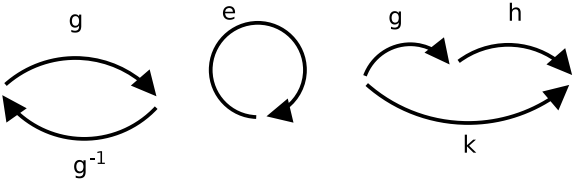
Groups

Examples of Groups:

- Rotation group
- Translation group
- Permutation group

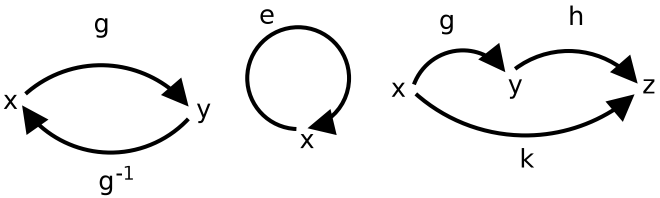
A Group G is a set where:

- Inverses: for all $g \in G \;\; \exists g^{-1} \in G$ s.t. $gg^{-1} = e$
- Identity element: $e \in G$ s.t. $\forall g \in G \;\; e \cdot g = g$
- Closure: For all $g, h \in G \, \exists k \in G$ s.t. $gh = k$



A group G acting on a set X:

- Notation: $G : X \to X$ with $g : x \mapsto y$ also written as $g \cdot x = y$ or $x \xrightarrow{g} y$
- Identity: $e \cdot x = x$
- Associative: $(g \cdot (h \cdot x)) = (g \cdot h) \cdot x$
- Invertable: $\left(g^{-1} \cdot (g \cdot x)\right) = \left(g^{-1} \cdot g\right) \cdot x = e \cdot x = x$ (non-dissipative)



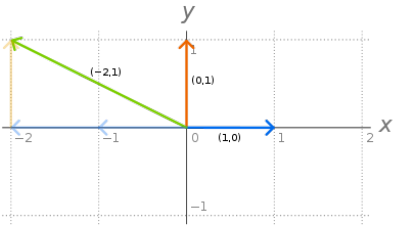
Vectors and Bases

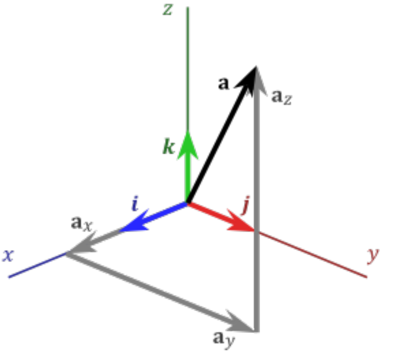
A Vector $\vec{v} \in \mathbb{R}^n$ in n-dimensional space is:

- A collection of n real numbers: $\vec{v} = \left(v^{(1)}, v^{(2)}, v^{(3)}, \cdots, v^{(n)}\right)$

A vector space basis for $\mathbb{R}^n$ is a collection of n vectors $\left\{\vec{e}_k : 1 \leq k \leq n\right\}$:

- Where $e_1 = (1, 0, 0, \cdots, 0)$ and $e_2 = (0, 1, 0, \cdots, 0)$ and $e_3 = (0, 0, 1, 0, \cdots, 0)$ and ...

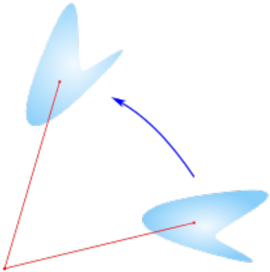


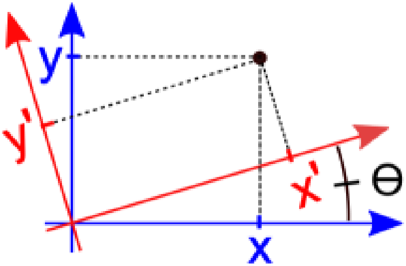


Passive and Active Rotations

- A rotation changes the direction of a vector: $\vec{x}' = R\vec{x}$
  - Body coordinates vs. Space coordinates
- A rotation can be represented by a matrix
  - In 2D:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$





Rotations in N dimensions

A rotation changes the direction of a vector: $\vec{x}' = R\vec{x}$

- In n dimensions :

$$
\begin{pmatrix} x^{(1)\prime} \\ x^{(2)\prime} \\ \vdots \\ x^{(n)\prime} \end{pmatrix}
=
\begin{bmatrix}
\cos\theta & 0 & \cdots & 0 & -\sin\theta & & \cdots & 0 \\
0 & 1 & & & 0 & & & 0 \\
\vdots & & \ddots & & \vdots & & & \\
0 & & & 1 & 0 & & & \vdots \\
\sin\theta & 0 & \cdots & 0 & \cos\theta & & & \\
 & & & & & 1 & & \\
\vdots & & & & & & \ddots & 0 \\
0 & 0 & & & \cdots & & 0 & 1
\end{bmatrix}
\begin{pmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(n)} \end{pmatrix}
$$

Infinitessimal Rotations

- An infinitessimal rotation:

$$\vec{x}+\delta\vec{x} = (I+\delta R)\vec{x} = \vec{x}+\delta R\vec{x} = \vec{x}+\left(\left.\frac{dR}{d\theta}\right|_{\theta=0}\delta\theta\right)\vec{x}$$

  - In 2D:

$$\vec{x} = \begin{pmatrix} x' \\ y' \end{pmatrix} = R\vec{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

  - but:

$$\left.\frac{d\cos\theta}{d\theta}\right|_{\theta=0} = 0 \quad \text{and} \quad \left.\frac{d\sin\theta}{d\theta}\right|_{\theta=0} = 1$$

  - so

$$\delta\vec{x} = \delta R\vec{x} = \left.\frac{dR}{d\theta}\right|_{\theta=0}\delta\theta\vec{x} = \delta\theta\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x} = \delta\theta L\vec{x}$$

  - The matrix L is called the “the infinitessimal generator of rotations” AKA “the angular momentum operator”.

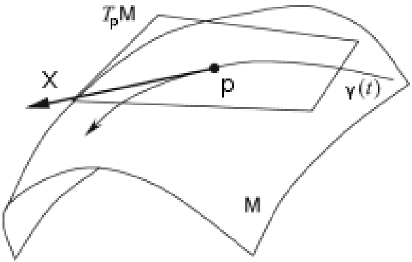
Partial derivatives

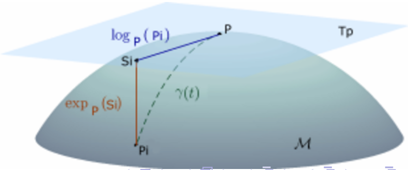
Given a curve $\gamma(t)$ in a manifold M, such that the curve is is tangent to the vector X at $p \in M$, the Lie derivative of a function f on M is:

$$\mathscr{L}_X f(p) = \left. \frac{f(\gamma(t)) - f(\gamma(0))}{t} \right|_{p=\gamma(0)} \text{ and } X=\gamma'(0)$$

Notation: the vector (field) X is written as

$$X = X^\mu \frac{\partial}{\partial x^\mu} = X^\mu \partial_\mu = X^\mu e_\mu$$

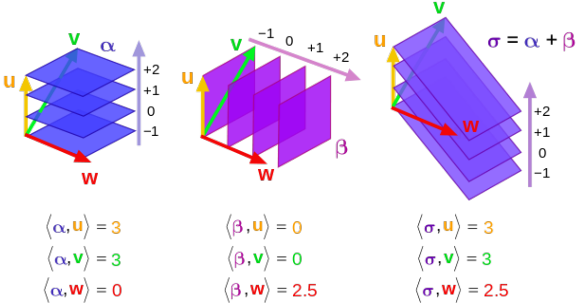




The dual basis: $e^{\mu}\left(e_{\nu}\right) = e^{\mu} e_{\nu} = \delta^{\mu}_{\nu}$

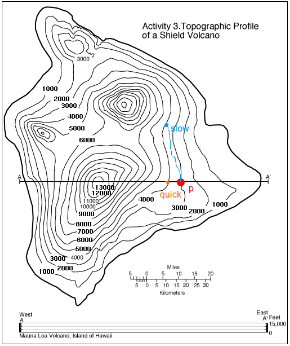
The Kronecker delta: $\delta^{\mu}_{\nu} = \begin{cases} 1 & \text{when } \mu = \nu \\ 0 & \text{when } \mu \neq \nu \end{cases}$

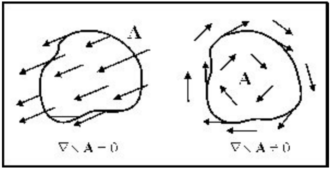
Partial deriviatives: $\partial_{\mu} = e_{\mu}$
Differential forms: $dx^{\mu} = e^{\mu}$
They are dual: $dx^{\mu}\left(\partial_{\nu}\right) = \delta^{\mu}_{\nu}$
A function that takes a vector and spits out a number (“counting surfaces”):

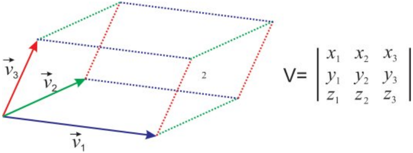


Examples of differential forms

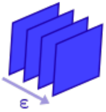
- The 1-form: df is like the gradient $\vec{\nabla} f$
- "Counting surfaces" are topographic contours (slices of const height)
- The 2-form: $dx \wedge dy$ is like the curl: $\vec{\nabla} \times \vec{v}$
- The 3-form $dx \wedge dy \wedge dz$ is like the volume determinant
- $\det I = \det [e_1, e_2, e_3]$

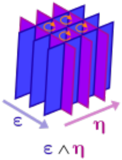


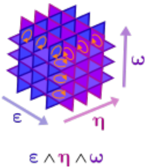




Wedge products







$$\varepsilon = \varepsilon_\mu dx^\mu$$

$$\eta = \eta_\mu dx^\mu$$

$$\varepsilon \wedge \eta = \varepsilon_\mu \eta_\nu dx^\mu \wedge dx^\nu$$

Antisymmetric: $dx \wedge dy = -dy \wedge dx$

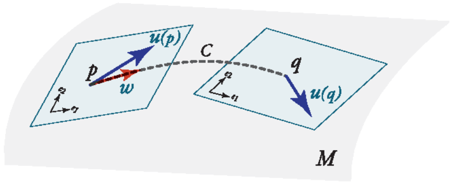
Linear: $adx + bdx = (a+b)\,dx$

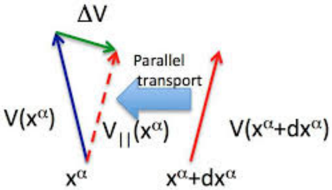
Tensorial: $T_{\mu\nu\cdots\rho} dx^\mu \wedge dx^\nu \wedge \cdots \wedge dx^\rho$

Covariant derivative

Joins neighboring fibers: $D = d + A$

- Alternate notation: $D^\mu = dx^\mu + A^\mu$ when moving in direction μ
- A is an infinitessimal rotation matrix: $A^\mu = [A^\mu]_{ij} = \Gamma^\mu_{ij}$
- Connection=Christoffel symbols
- Fiber coordinates: index i, j act on the fiber
- Base space coordinates:μ is a direction in the base space.





Curvature

Moving (alternately) in two directions:

- Notation: Field strength 2-form: $F = D \wedge D = dA + A \wedge A$
- Notation: Curvature tensor:
  $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - [X,Y]$

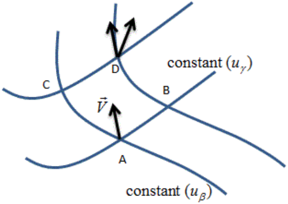
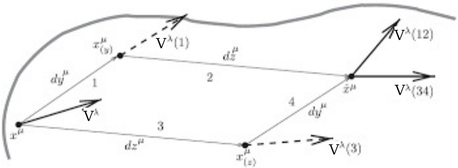


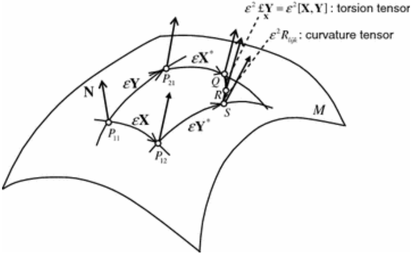
Fig 1.2

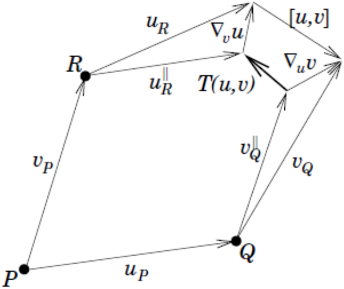


Torsion

Moving (alternately) in two directions:

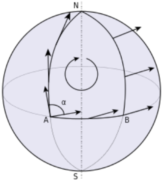
- Notation: Torsion form: $\Theta = D\theta = d\theta + A \wedge \theta$
- ... where θ is the solder form: $\theta = \sum_i p_i dq_i$
- Notation: Torsion tensor: $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$
- There is one unique torsionless connection: the Levi-Civita connection

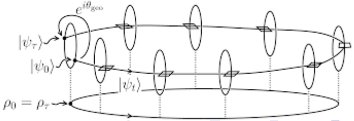




E&M is (just) a circle bundle!

- Each group element: $g = e^{i\theta}$= coordinates on circle
- Vector potential: $A_\mu = \left(\vec{A}, \phi\right)$
- Choice of gauge = $A_\mu \to A'_\mu = A_\mu + g^{-1}\partial_\mu g$
- Curvature $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$
- ...or $\vec{E} = \vec{\nabla}\phi$ and $\vec{B} = \vec{\nabla} \times \vec{A}$
- Geodesics go "splat" on an electric charge!
- Holonomy is the Bohm-Aharonov effect!

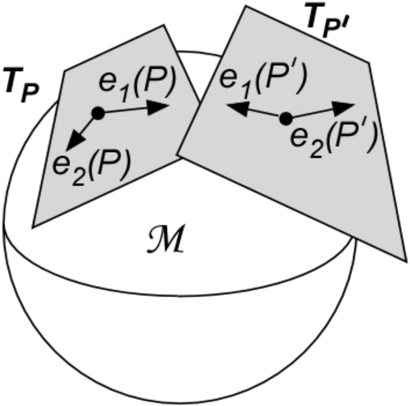




Frame fields (Vierbeins)

A frame field is:

- A collection of basis vectors $\{\vec{e}_k(p)\}$
- One for each point $p \in U_\alpha$



Metric Tensor

The metric is (just) a product of vierbeins (frames)

$$g_{\mu\nu} = e_\mu \cdot e_\nu = e^a_\mu \eta_{ab} e^b_\nu$$

where $\eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Gives the length of a vector:

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v^a v^b \eta_{ab}} = \sqrt{\left(v^{(x)}\right)^2 + \left(v^{(y)}\right)^2 + \left(v^{(z)}\right)^2 - \left(v^{(t)}\right)^2}$$

General Relativity

General Relativity is (just) a frame bundle!

- Each group element: $g \in SO(3,1)$
- Connection: $A_\mu = \Gamma_\mu^{\rho\sigma}$
- Curvature: $F = dA + A \wedge A$
- ... that is, curvature $R_{\mu\nu}^{\rho\sigma} = \partial_\mu A_\nu^{\rho\sigma} - \partial_\nu A_\mu^{\rho\sigma} + \frac{1}{2}\left[A_\mu, A_\nu\right]^{\rho\sigma}$
- Choice of gauge == choice of coordinate frame!
- Choice of gauge = $A_\mu \to A'_\mu = A_\mu + g^{-1}\partial_\mu g$
- Geodesics go “splat” on a black hole singularity!

Conclusion

Unification of Physics

- Fiber bundles unify all of the fundamental physics theories
- So what is there left to unify?
- Well, why/how $U(1) \times SU(2) \times SU(3) \times SO(3,1)$?
- Kaluza-Klein theory (the 5-sphere)
- Affine Lie groups (string theory)
- Supersymmetry (fermions)

Placeholder

Geometry with Formulas

The hardest part with formulas is (1) there are so many (2) there are many different ways of writing down the **same** equations, using wildly different notation.

- Introduce Lie derivative $L_X f$
- Introduce covariant derivative D = d+A – Rosetta stone of different notations
- Geodesics as solutions of Hamilton's equations i.e. as linear, first-order diffeq NOT second order!

$$\dot{p} = -\frac{dH}{dq} \qquad \dot{q} = \frac{dH}{dp}$$

  where H=squared-length-of-curve
- exp as the map that moves along geodesics
- Geodesic completeness

Metric Differential Geometry

- Indroduce metric as inner product of frame fields $g_{\mu\nu} = e_\mu \cdot e_\nu = e_\mu^{\ a} e_\nu^{\ b} \eta_{ab}$
- Metric was NOT needed to define curvature, geodesics, parallel transport
- (metric is almost kind-of useless except that its a standard touch-stone for GR)
- Provide (repeat) Einstein eqns.
- Replace frame field by generic fiber bundle
- e.g. U(1) for electromagnetism, SU(n) for yang-mills
- Maxwell's equations are nothing more than Hamilton's eqns on U(1) + Bianchi identites

$$F = dA \quad d * F = 0$$

- Yang-Mills/Einstein

$$F = dA + A \wedge A \qquad D * F = 0$$

is the same as

Geodesics

- Maxwell's eqn's have singularities called "electric charges" and geodesics go "splat" on an electric charge
- Swarzschild BH's are just like electric charges: geodesics go splat when they get there.